Tens

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By trying small values of n, it seems that all 4 expressions evaluate to some multiple of 10. I have three different methods for proving this.

Method 1: using modular arithmetic

This method is effective for proving divisibility by 10 for the four expressions, but it doesn't really help us explain them, nor does it help us find similar results.

$9^n + 1^n$ for odd n

If n is an odd integer, then let $n = 2m + 1$ for some integer m.

$$9^{2m+1} \equiv 9 \times 9^{2m} \equiv 9 \times 81^m \equiv 9 \times 1^m \equiv 9 \ (mod\ 10)$$

Hence $9^n + 1^n \equiv 9 + 1 \equiv 10 \equiv 0\ (mod\ 10)$ for odd n, which is equivalent to saying that this sum is divisible by ten for odd n.

$7^n + 3^n$ for odd n

As before, let $n = 2m + 1$.

$$7^{2m+1} \equiv 7^{2m} \times 7 \equiv 49^m \times 7 \equiv 9^m \times 7\ (mod\ 10)$$

$$3^{2m+1} \equiv 3^{2m} \times 3 \equiv 9^m \times 3\ (mod\ 10)$$

Hence for odd n, $7^n + 3^n \equiv 9^m \times 7 + 9^m \times 3 \equiv 9^m(7+3) \equiv 9^m \times 10 \equiv 0\ (mod\ 10)$, so that $7^n + 3^n$ is divisible by 10 for odd n.

$8^n - 2^n$ for even n

If n is even, then let $n = 2m$ for some integer m.

$$8^{2m} - 2^{2m} \equiv 64^m - 4^m \equiv 4^m - 4^m \equiv 0\ (mod\ 10)$$

So $8^n - 2^n$ is divisible by 10 for even n.

$6^n - 4^n$ for even n

$$6^{2m} - 4^{2m} \equiv 36^m - 16^m \equiv 6^m - 6^m \equiv 0\ (mod 10)$$

So $6^n - 4^n$ is divisible by 10 for even n

Method 2:using Binomial expansion

I shall only give an outline of this method. It does better at explaining the divisibility, but I prefer method 3.

For the first two expressions, a general form could be written as $x^{2m+1} + (10-x)^{2m+1}$. Expanding the second term,

$$x^{2m+1} + (10-x)^{2m+1} = x^{2m+1} + \left(10^{2m+1} - \binom{2m+1}{1}10^{2m}x + \cdots \binom{2m+1}{2m}10x^{2m} - x^{2m+1}\right)$$

Note that in the brackets on the RHS, all of the terms are divisible by ten except the $-x^{2m+1}$. This is resolved by the addition of x^{2m+1} outside the brackets, meaning that the LHS expression is divisible by ten. A similar proof can be written for the last two expressions by generalising them to $x^{2m}-(10-x)^{2m}$.

Method 3: factorising

By factorising generalised expressions we can not only prove all four of the given expressions are divisible by ten, but also find many others with the same property. For example,

$$x^{2m+1}+y^{2m+1}=(x+y)(x^{2m}-x^{2m-1}y+x^{2m-2}y^2-\cdots-xy^{2m-1}+y^{2m})$$

Meaning that for any two numbers x and y that add up to 10 (or any multiple of 10), x^n+y^n is divisible by 10 for odd n. Thus this proves the first two given expressions divide by ten. Secondly, observe that

$$x^{2m}-y^{2m}=(x+y)(x^{2m-1}-x^{2m-2}y+\cdots-y^{2m-1})$$

Meaning that for any two numbers that add up to some multiple of 10, x^n-y^n is divisible by 10 for even n. This factorisation therefore proves the last two expressions divide by ten.

This method is slightly similar to method 2, but I think it is superior since it is more explicit and more readily suggests further results. For example, a different set of results can be found by considering those generalised expressions that have a factor of $(x-y)$. I tried all of the possible combinations of $+/-$ and odd and even n, and arrived at two factorisations:

$$x^{2m}-y^{2m}=(x-y)(x^{2m-1}+x^{2m-2}y+\cdots+y^{2m-1})$$

And

$$x^{2m+1}-y^{2m+1}=(x-y)(x^{2m}+x^{2m-1}y+x^{2m-2}y^2+\cdots+xy^{2m-1}+y^{2m})$$

These can of course be combined more simply into

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\cdots+y^{n-1})$$

So for any two numbers x and y with a difference of some multiple of 10, x^n-y^n is also divisible by 10 for all positive integers n. If you think about it, this is intuitive; you would expect 13^n-3^n to end in a zero, since you would imagine the 3s to 'cancel each other out'.