All Tangled Up

The first thing to notice is how you can chain 'R' and 'T' together.

→ R can only be used once in a row: twice will get you to the initial fraction $\left(\frac{a}{b} \xrightarrow{R} -\frac{b}{a} \xrightarrow{R} \frac{a}{b}\right)$

→ T can be used as many time as required, and T^n will add n e.g. $0 \xrightarrow{T^{\frac{n+1}{2}}} \frac{n+1}{2}$

→ For fractions : $\frac{a}{b} \xrightarrow{T^n} \frac{a+nb}{b}$

→ The inverse of T is $x-1$; inverse $R=R$

I've used $-T$ to signify this

(1) To find $0 \rightarrow \frac{n}{n+1}$, it's useful to work backwards:

$$\frac{n}{n+1} \xrightarrow{-T} \frac{n}{n+1} - \frac{n+1}{n+1} = \frac{-1}{n+1} \xrightarrow{R} n+1 \xrightarrow{-T^n} 0$$

So T^nRT is $\frac{n}{n+1}$

(2) Testing out T^2RT, T^2RT^2RT etc. you get $\frac{1}{2}, \frac{1}{3} \dots \frac{1}{n}$

To prove this, I split up T^2RT^2RT into $T(TRT)(TRT)$, and tested TRT on $\frac{1}{n}$: $\xrightarrow{T} \frac{n+1}{n} \xrightarrow{R} \frac{-n}{n+1} \xrightarrow{T} \frac{-n}{n+1} + \frac{n+1}{n+1} = \frac{1}{n+1}$

$1 = \frac{1}{1}$ is $\frac{1}{n}$. You need an extra T at the start to ensure n isn't 0.

Each time you apply TRT, n increases by 1, so $\frac{1}{10} = T(TRT)^9$

Repeat 9 times

③ Now, to both find out if it was possible to create any fraction from 0 and find more patterns, I wrote out how to get to a range of fractions which were between : $0 < x \leq 1$, since if they were greater (or less than), you could use T to add 1 to a fraction between $0 < x \leq 1$

$\frac{1}{2}, \frac{2}{2} = T^2RT, T$

$\frac{1}{3}, \frac{2}{3}, \frac{3}{3} = T(TRT)^2, T^3RT, T$
T^2RT^2RT

$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = T(TRT)^3, T^2RT, T^4RT, T$

$\frac{1}{5}, \frac{2}{5} \dots \frac{5}{5} = T(TRT)^4, T^3RT^2RT, T^2RT^3RT, T^5RT, T$

$\frac{1}{6}, \frac{2}{6} \dots \frac{6}{6} = T(TRT)^5, T^2RT^2RT, T^2RT, T^3RT, T^6RT, T$

$\frac{1}{7}, \frac{2}{7} \dots \frac{7}{7} = T(TRT)^6, T^3RT^2RT^2RT, T^4RT^2RT, T^2RT^2RT^3RT, T^2RT^4RT,$
T^7RT, T

- This allows us to spot some patterns. We can already see $\frac{1}{n}$ and $\frac{n}{n+1}$ (or $\frac{n-1}{n}$), which we have proved earlier.

- However, for $\frac{n-2}{n}$, there's a difference between when n is odd and even. For every even 'n', the combinations slightly change:

$\left(\frac{2}{4} \to \frac{4}{6}\right) = T^2RT \to T^3RT$, the first T increases by 1.
↳ To find this 'power' of T, you do $\frac{n}{2}$

- This gives you $T^{\frac{n}{2}}RT = \frac{n-2}{n}$

Proof this works for all even numbers.

$$0 \xrightarrow{T^{\frac{n}{2}}} \frac{n}{2} \xrightarrow{R} \frac{-2}{n} \xrightarrow{T} \frac{n-2}{n}$$

However this only works for even numbers, as otherwise, $\frac{n}{2}$ would not give a whole number, which is required (T can only be performed a whole number of times).

For every odd 'n', the combination also slightly changes:

$\left(\frac{1}{3} \rightarrow \frac{3}{5} \rightarrow \frac{5}{7}\right) = T^2R\underline{\underline{T}}^2RT \rightarrow T^2R\underline{\underline{T}}^3RT \rightarrow T^2R\underline{\underline{T}}^4RT$

- Here's how the power on the second T set relate:

power $\{$ 2 3 4 $\rightarrow$ To get from the number (n), to the power, put it in $\frac{n+1}{2}$

number $\{$ 3, 5, 7

Proof this works for all odd numbers

$$0 \xrightarrow{T^2R} -\frac{1}{2} \xrightarrow{T^{\frac{n+1}{2}}} \frac{-1}{2} + \frac{n+1}{2} = \frac{n}{2} \xrightarrow{RT} \frac{n-2}{n}$$

However, this method only works for $\frac{n-2}{n}$. Ideally, we would be able to perform $0 \xrightarrow{T^{\frac{n}{m}}} \frac{n}{m} \xrightarrow{R} \frac{-m}{n} \xrightarrow{T} \frac{n-m}{n}$

but $\frac{n}{m}$ isn't always a whole number.

If you look again at the fractions, you can notice something else

$\frac{n-2}{n}$ $\quad \frac{3}{5} = T^2RT^3RT = (\underbrace{T^2RT^2}_{3/2})(TRT)$

$\frac{n-3}{n}$ $\quad \frac{1}{4} = T^2RT^2RT^2RT = (\underbrace{T^2RT^2RT}_{1/3})(TRT)$

$\frac{n-4}{n}$ $\quad \frac{3}{7} = T^4RT^2RT = (\underbrace{T^4RT}_{3/4})(TRT)$

- Each set of functions can be split into 2 parts: Part 2 = 'TRT' and Part 1 = $\frac{n-m}{m}$, where m as in '$\frac{n-4}{4}$'
- You can split Part one further, into a fractional component and a whole number component e.g.

$\frac{n-3}{n}$ $\quad \frac{4}{7} = T^2RT^2RT^3RT = (\underbrace{T^2RT^2RT}_{1/3})(\underbrace{T}_{+1})(TRT)$

Whole number components can easily be added by adding 'T' → To prove that you can make a set of $\frac{n-m}{n}$ fractions, you must prove that you can make $\frac{1}{m}, \frac{2}{m}, \frac{3}{m} \ldots \frac{m}{m}$

Let's have a look at $\frac{1}{2}$:

$\frac{n-1}{n}$ $\quad \frac{1}{2} = T^2RT$. Because you can make $\frac{1}{2}$, you have proved that you can make any fraction $\frac{n-2}{n}$, because the fractional remainder of $\frac{n-2}{2}$ is always ½ (or 0/1)

- Using this, you can show that you can make $\frac{1}{3}$: $\left(\frac{3-2}{3} = \frac{1}{3}\right)$. This allows you to create any fraction $\frac{n-3}{n}$, as fractional remainders of $\frac{n-3}{3}$ are $\frac{1}{3}$, $\frac{2}{3}$ (or 0/1).

Proved since all fractions $\frac{n-1}{n}$ are proved, as shown earlier.

- This proves $\frac{1}{4}$ as $\frac{n-3}{n}$ is possible, allowing the cycle to continue.

Overall

- To prove you can make any fraction, you must prove that you can make any $\frac{n}{m}$ as $\frac{n}{m}$ R $\frac{n-m}{n}$
- $\frac{n}{m}$ can be split up into an integer and a fraction
  → Integers can be added by using 'T'
  → You must prove you can make the fraction, which will be from $\frac{1}{m}, \frac{2}{m} \ldots \frac{m}{m}$
- You can make $\frac{n}{1}$ using T^n → this proves you can have $m=1$ and make $\frac{n-1}{n}$, or $\frac{2-1}{2} = \frac{1}{2}$
- Because you know how to make ½, you can make any $n/2$ because the only fractional remainder is ½
  → $m=2$; $\frac{n-2}{n}$ → can make $\frac{3-2}{3} = \frac{1}{3}$ and $\frac{3-1}{3} = \frac{2}{3}$

- Using this, you can make all possible fractional components of $\frac{n}{3}$ $\left(\frac{1}{3} \text{ and } \frac{2}{3}\right)$ proving $\frac{n-3}{n}$
- This proof proves even more fractional components, allowing $\frac{n-4}{n}$

→ Continues, proving $\frac{n-n}{n}$

- Proof of $\frac{n-n}{n}$ means you can make any fraction from $\frac{1}{n}$ to $\frac{n}{n}$
- Adding T therefore allows you to create any positive fraction
- This means that by using R, you can find the negative reciprocal of a fraction. If you know all positive fractions, you know all negative fractions.