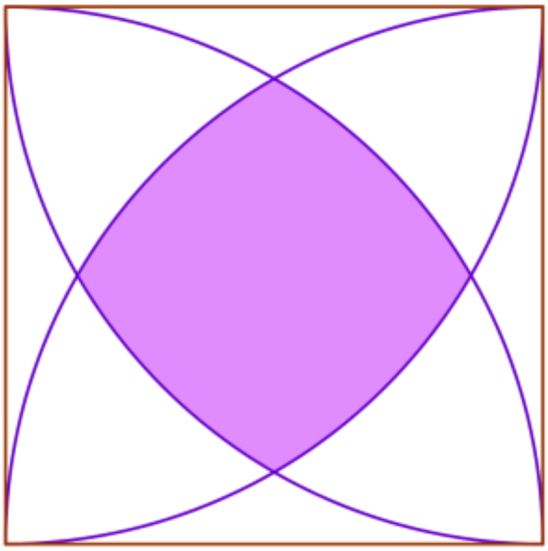
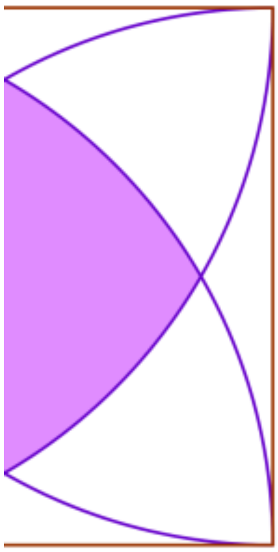
Area of a curved square (calculus method)



First, we split the diagram in half, cutting it vertically:



To find half the shaded area here (a quarter of the area of the curved square), we need to find the integral of the curve running from the top of the curved triangle shown to the bottom right corner of the unit square, and subtract the area of the rectangle bounded by the following four points:

(0.5, 0)
(tip of triangle, 0)
(0.5, 0.5)
(tip of triangle, 0.5)

The curve, going from the top of the curved triangle to the bottom right corner of the square has the equation of $y = +\sqrt{1 - x^2}$ (since a circle radius 1 around the origin has the equation of $x^2 + y^2 = 1$, and since the arc is above the x axis).

Now that we have the equations of the curve, we need to establish the limits of the definite integral. The lower limit is simple; it's just $\frac{1}{2}$, as the square has a side length of 1, and the axis that bounds the curved triangle is halfway along the square, at $x = \frac{1}{2}$. The upper limit is the x coordinate of the point of intersection between two of the curves on the diagram. This is also easy to find, as it lies on the line $y = \frac{1}{2}$. Substituting the value of $\frac{1}{2}$ into the equation $y = +\sqrt{1 - x^2}$, we get $x = \frac{\sqrt{3}}{2}$, or roughly 8.66, hence the area of the shaded curved triangle shown will be:

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1 - x^2}\ dx$$

The integral works out to be $\frac{1}{2}\left(\sqrt{1 - x^2 +} + x + \sin^{-1} x\right)$ for the limits of $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$, or approximately 0.262.

The rectangle's area is $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \cdot \frac{1}{2}$, or roughly 0.183. Subtracting this from the integral of the curve, we get approximately 0.079.

Finally, we multiply this by 4 to get the full area of the curved square, which works out to be 0.315, to the accuracy of 3 significant figures.

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