Giants

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As the numbers get larger and larger, simply testing whether their difference is positive or negative soon goes beyond capabilities of my calculator. What is required is a more general approach. My approach not only works for large numbers, but can also be applied to smaller numbers too.

Let $x^y > y^x$

$$x^y > y^x \Leftrightarrow \ln x^y > \ln y^x \Leftrightarrow y \ln x > x \ln y \Leftrightarrow \frac{\ln x}{x} > \frac{\ln y}{y}$$

Let $f(n) = \frac{\ln n}{n}$. Hence $f(x) > f(y) \Leftrightarrow x^y > y^x$.

Differentiating $f(n)$ with respect to n using the Quotient Rule,

$$f'(n) = \frac{n \times {}^1/_n - \ln n \times 1}{n^2} = \frac{1 - \ln n}{n^2}$$

Hence the function has a stationary point at $\ln n = 1$, i.e. at $n = e$, $f(n) = e^{-1}$, and no others.

More importantly, it shows that $f(n)$ is a decreasing function for $n > e$, since this is where $f'(n) = \frac{1-\ln n}{n^2}$ is negative.

Hence

$$e \leq x < y \Rightarrow f(x) > f(y)$$

And so

$$e \leq x < y \Rightarrow x^y > y^x$$

So

$$9^{10} > 10^9,$$

$$99^{100} > 100^{99},$$

$$999^{1000} > 1000^{999},$$

$$\ldots,$$

$$\{a\ billion\ 9s\}^{\{1\ and\ a\ billion\ 0s\}} > \{1\ and\ a\ billion\ 0s\}^{\{a\ billion\ 9s\}},$$

$$\ldots,$$