Common Divisor

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Part A

$1^3-1,\ 2^3-2,\ 3^3-3,\ 4^3-4, \ldots, n^3-n$

$\{=0, 6. 24, 60, n^3-n\}$

$n^3-n = n^2-1 \cdot n$

$=(n+1)(n-1)(n)$

since $n-1, n, n+1$ are 3 consecutive integers, one must be a multiple of 3, furthermore at least one must be a multiple of 2. Therefore the largest is 3×2 which is 6.

Part B

$1^5-1^3,\ 2^5-2^3,\ 3^5-3^3,\ 4^5-4^3, \ldots, n^5-n^3$

$\{=0, 24, 216, 960, n^5-n^3\}$

$n^5-n^3 = n^3(n^2-1)$

$=n^3(n+1)(n-1)$

$(n-1)\cdot n\cdot n\cdot n\cdot(n+1)$

If n is odd, then both $n-1$ and $n+1$ will be even meaning they are divisible by 2, this will then mean that n must be divisible by 3. So we have 3 factors of 2 and a factor of 3. This means the lowest divisor of n^5-n^3 is $2^3\times3 = 8\times3 = 24$.

Part C

$1^5-1,\ 2^5-2,\ 3^5-3, \ldots, n^5-n$

$\{=0, 30, 240, \ldots, n^3-n\}$

n^5-n

$=n(n^4-1) = n(n^2+1)(n^2-1)$

$=n(n^2+1)(n-1)(n+1).$

$n^5 - n = n(n^4 - 1)$

$n(n^2-1)(n^2+1)$

$n(n+1)(n-1)(n^2+1)$

Since $n, n+1$ are two consecutive integers, it follows that at least one is a even number, while the other being odd. Similarly, $n-1, n, n+1$ are 3 consecutive integers, this meaning one is a multiple of 3.

To show that $n^5 - n$ is divisible by 5 we can use proof by induction

Assume the base case $n = k$

$$(k^5 - k) \equiv 0 \pmod 5$$

meaning $k^5 - k = 5n$

therefore if its true it must hold for $n = k+1$

$(k+1)^5 - (k+1)$

$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k+1)$

$= k^5 + 5k^4 + 10k^3 + 10k^2 + 4k$

$(k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$

$(k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$

Since the second part is divisible by 5 for the case $n = k+1$ to be true, $k^5 - k$ must be divisible by 5

So the highest divisior is the 3 prime factors multiplied together, so $5 \times 3 \times 2 = 30$, therefore 30 is the highest number which can be divided into all the terms.

Part D.

$2^{2n}-1 \ \% \ 3 = 0$, for all n

we have two ways to counter this problem,

OPTION 1:

$2^{2n}-1 = (2^n+1)(2^n-1)$

since $2^n-1, 2^n, 2^n+1$ are 3 consecutive integers, and 2^n cannot be divided by 3, one of 2^n+1 or 2^n-1 can be divided by 3.

If one of them is divisible by 3 then the whole product is divisible by 3.

OPTION 2:

$2^{2n}-1 \equiv 4^n-1$

we can use the expansion of $x^n - y^n$

$= (x-y)(x^{n-1} + x^{n-2}y + \dots xy^{n-2} + y^{n-1})$

or

$(x-y) \cdot \sum_{k=0}^{n-1} b^k a^{n+(k-1)}$. or $(x-1) \cdot \sum_{k=0}^{n-1} b^k$.

$4^n - 1 = (4-1)(4^{n-1} + 4^{n-2} + 4^{n-3} + \dots + 4^0)$

$4^n - 1 = (4 \cdot 4^{n-1} + 4 \cdot 4^{n-2} + 4 \cdot 4^{n-3} + 4 \cdot 4^0) \dots$

$- 1 \cdot 4^{n-1} - 1 \cdot 4^{n-2} - 1 \cdot 4^{n-3} + (-1) \cdot 4^0) \dots$

$= (3 \cdot 4^{n-1} + 3 \cdot 4^{n-2} + 3 \cdot 4^{n-3} + \dots 3 \cdot 4^0)$

$= 3(4^{n-1} + 4^{n-2} + 4^{n-3} \dots + 4^0)$

$\therefore$ It is divisible by 3