a)

$1^3 - 1 = 0$

$2^3 - 2 = 6$

$3^3 - 3 = 24$

$4^3 - 4 = 60$

$n^3 - n = \underbrace{(n-1)n(n+1)}$

this is a list of three consecutive numbers, so exactly one of them is divisible by 3, and at least one is divisible by 2. These are co-prime, so we know that their product $(2 \times 3 =)\ 6 \mid n^3 - n \ \forall\, n \in \mathbb{Z}$.

b)

$1^5 - 1^3 = 0$

$2^5 - 2^3 = 24$

$3^5 - 3^3 = 216$

$4^5 - 4^3 = 960$

$$n^5 - n^3 = n^3(n^2-1)$$
$$= n^3(n+1)(n-1)$$

As before 3 divides one of $n-1, n, n+1$, and therefore one of $n-1, n^3, n+1$.

If n is even, $8 | n^3$. If n is odd, either $n \equiv 1$ [mod 4] or $n \equiv 3$ [mod 4].

In either case, one of $(n-1), (n+1)$ is divisible by 4 and the other is divisible by 2. So $(n-1)n^3(n+1)$ is necessarily divisible by 8.

As 3 is co-prime to 8, their product $24 | (n-1)n^3(n+1)$.

c)
$$n^5 - n = n(n^4 - 1)$$
$$= (n-1)n(n+1)(n^2+1)$$

From as, $6 | (n-1)n(n+1)$. Consider the following cases:

$n \equiv 0$ [mod 5], $5 | n$ $\quad$ $n \equiv \pm 2$ [mod 5], $5 | n^2+1$

$n \equiv 1$ [mod 5], $5 | n-1$ $\quad$ $n \equiv 4$ [mod 5], $5 | n^2+1$.

In all cases $5 \mid (n-1)n(n+1)(n^2+1)$.

$6 \perp 5$, So $6 \times 5 = 30 \mid (n-1)n(n+1)(n^2+1)$.

d)

$$2^{2n} - 1 = (2^n)^2 - 1$$

$$\equiv ((-1)^n)^2 - 1 \quad [\text{mod } 3]$$

$$\equiv (\pm 1)^2 - 1 \quad [\text{mod } 3]$$

$$\equiv 0 \quad [\text{mod } 3]$$

So $3 \mid 2^{2n} - 1$ for $n \in \mathbb{N}$.