COMMON DIVISOR

PART A

$1^3 - 1 = 0$
$2^3 - 2 = 6$
$3^3 - 3 = 24$
$4^3 - 4 = 60$

Div. by 6?

$$n^3 - n = n(n^2 - 1)$$
$$= n(n-1)(n+1)$$
$$= (n-1)(n)(n+1)$$

This is the product of 3 consecutive integers

If there are 3 consecutive integers, then at least one of them will be div. by 3.

Also in any 2 consecutive integers, one of the two will be even, i.e. div by 2

Thus these are div. by 2 and 3 $\Rightarrow$ $\boxed{\text{div. by } 6}$

PART B

$1^5 - 1^3 = 0$
$2^5 - 2^3 = 24$
$3^5 - 3^3 = 216$
$4^5 - 4^3 = 960$

Div. by 24?

Divisible by 3

$$n^5 - n^3 = (n-1)(n+1)(n)(n^2)$$

a) If $n = 2k$

Div by 8

$$n^5 - n^3 = (2k-1)(2k+1)(2k)(4k^2)$$

b) If $n = 2k+1$

$$n^5 - n^3 = (2k)(2k+2)(2k+1)^3$$
$$= 4(k)(k+1)(2k+1)^3$$

2 consecutive always div. by 2

div. by 4

$\therefore$ Divisible by 8

> Always divisible by 3 and 8
>
> $\Rightarrow$ divisible by 24

PART C

$$1^5 - 1 = 0$$
$$2^5 - 2 = 30$$
$$3^5 - 3 = 240$$
$$4^5 - 4 = 1020$$

By 30?

$$n^5 - n = n(n^4 - 1)$$
$$= n(n^2-1)(n^2+1)$$
$$= \underbrace{n(n-1)(n+1)}_{\text{div. by } 6}(n^2+1)$$

Using the hint and this observation, the expression will perhaps be divisible by 5 as well

✓ If $n = 5k$, Exp. $= (5k)(5k-1)(5k+1)((5k)^2+1)$

div by 5

✓ If $n = 5k+1$, Exp. $= (5k+1)(5k)(5k+2)((5k+1)^2+1)$

div by 5

✓ If $n = 5k+2$, Exp. $= (5k+2)(5k+1)(5k+3)[(5k+2)^2+1]$

$= (5k+2)(5k+1)(5k+3)[25k^2+10k+4+1]$

$= (5k+2)(5k+1)(5k+3)[25k^2+10k+5]$

$= (5k+2)(5k+1)(5k+3)(5)[5k^2+2k+1]$

div by 5

✓ If $n = 5k+3$, Exp. $= (5k+3)(5k+2)(5k+4)[(5k+3)^2+1]$

$= (5k+3)(5k+2)(5k+4)[25k^2+15k+9+1]$

$= (5k+3)(5k+2)(5k+4)[25k^2+15k+10]$

$= (5k+3)(5k+2)(5k+4)(5)[5k^2+3k+2]$

div by 5

✓ If $n = 5k+4$, Exp. $= (5k+4)(5k+3)(5k+5)((5k+4)^2+1)$

$= (5k+4)(5k+3)(5)(k+1)((5k+4)^2+1)$

div by 5

In all cases of $n = 5k + r$, $0 \leq r < 5$ the expression is divisible by 5 and 6

$\boxed{\text{Thus it is also divisible by } 30}$.

PART D

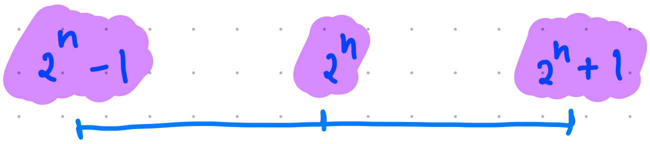
$2^2 - 1 = 3$

$2^4 - 1 = 15$

$2^6 - 1 = 63$

$\vdots$

$$2^{2n} - 1 = (2^n)^2 - (1)^2$$
$$= (2^n - 1)(2^n + 1)$$



The three numbers are consecutive

☆ It is certain that 2^n will never be divisible by 3 since it doesn't have a factor of 3

But in three consecutive numbers, one of the three will always be divisible by 3

$\Rightarrow$ Either of $2^n - 1$ OR $2^n + 1$ will be divisible by 3

$\Rightarrow$ Their product will also be divisible by 3

PART D is really elegant !!

The following 3 pages have the same solution but without the dotted template that might make it a bit hard to see, maybe? I realised later that it could be changed.

COMMON DIVISOR

PART A

$$1^3-1 = 0$$
$$2^3-2 = 6$$
$$3^3-3 = 24$$
$$4^3-4 = 60$$

Div. by 6?

$$n^3-n = n(n^2-1)$$
$$= n(n-1)(n+1)$$
$$= (n-1)(n)(n+1)$$

This is the product of 3 consecutive integers

If there are 3 consecutive integers, then at least one of them will be div. by 3.

Also in any 2 consecutive integers, one of the two will be even, i.e. div by 2

Thus these are div. by 2 and 3 $\Rightarrow$ $\boxed{\text{div. by } 6}$

PART B

$$1^5-1^3 = 0$$
$$2^5-2^3 = 24$$
$$3^5-3^3 = 216$$
$$4^5-4^3 = 960$$

Div. by 24?

$$n^5-n^3 = (n-1)(n+1)(n)(n^2)$$

Divisible by 3

a) If $n = 2k$

$$n^5-n^3 = (2k-1)(2k+1)(2k)(4k^2)$$

Div by 8

b) If $n = 2k+1$

$$n^5-n^3 = (2k)(2k+2)(2k+1)^3$$
$$= 4(k)(k+1)(2k+1)^3$$

2 consecutive always div. by 2

div. by 4

$\therefore$ Divisible by 8

Always divisible by 3 and 8

$\Rightarrow$ divisible by 24

PART C

$$1^5 - 1 = 0$$
$$2^5 - 2 = 30$$
$$3^5 - 3 = 240$$
$$4^5 - 4 = 1020$$

By 30?

$$n^5 - n = n(n^4 - 1)$$
$$= n(n^2-1)(n^2+1)$$
$$= n(n-1)(n+1)(n^2+1)$$

div. by 6

Using the hint and this observation, the expression will perhaps be divisible by 5 as well

✓ If $n = 5k$, $\text{Exp.} = (5k)(5k-1)(5k+1)((5k)^2+1)$

div by 5

✓ If $n = 5k+1$, $\text{Exp.} = (5k+1)(5k)(5k+2)((5k+1)^2+1)$

div by 5

✓ If $n = 5k+2$, $\text{Exp.} = (5k+2)(5k+1)(5k+3)[(5k+2)^2+1]$

$$= (5k+2)(5k+1)(5k+3)[25k^2+10k+4+1]$$
$$= (5k+2)(5k+1)(5k+3)[25k^2+10k+5]$$
$$= (5k+2)(5k+1)(5k+3)(5)[5k^2+2k+1]$$

div by 5

✓ If $n = 5k+3$, $\text{Exp.} = (5k+3)(5k+2)(5k+4)[(5k+3)^2+1]$

$$= (5k+3)(5k+2)(5k+4)[25k^2+15k+9+1]$$
$$= (5k+3)(5k+2)(5k+4)[25k^2+15k+10]$$
$$= (5k+3)(5k+2)(5k+4)(5)[5k^2+3k+2]$$

div by 5

✓ If $n = 5k+4$, $\text{Exp.} = (5k+4)(5k+3)(5k+5)((5k+4)^2+1)$

$$= (5k+4)(5k+3)(5)(k+1)((5k+4)^2+1)$$

div by 5

In all cases of $n = 5k + r$, $0 \leq r < 5$ the expression is divisible by 5 and 6

$$\boxed{\text{Thus it is also divisible by } 30}$$

PART D

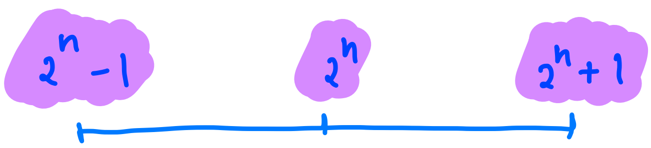
$2^2 - 1 = 3$

$2^4 - 1 = 15$

$2^6 - 1 = 63$

$\vdots$

$$2^{2n} - 1 = (2^n)^2 - (1)^2$$
$$= (2^n - 1)(2^n + 1)$$



The three numbers are consecutive

☆ It is certain that 2^n will never be divisible by 3 since it doesn't have a factor of 3

But in three consecutive numbers, one of the three will always be divisible by 3

$\Rightarrow$ Either of $2^n - 1$ OR $2^n + 1$ will be divisible by 3

$\Rightarrow$ Their product will also be divisible by 3

PART D is really elegant!!