The following solution is divided into sub-sections for the purpose of clarity. It is recommended that you read all the parts in chronological order.

Initial Ideas

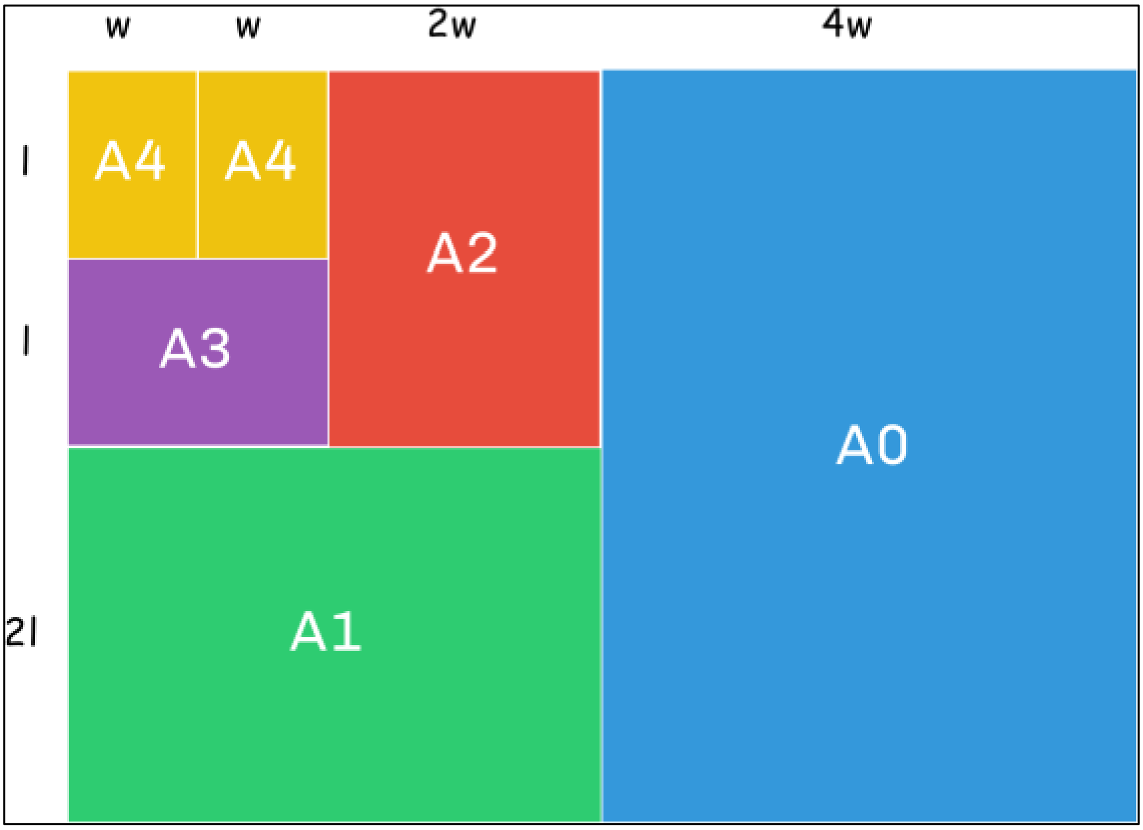
The problem presents an interesting application of mathematics in the real world – most of us use these A family papers in our daily lives either for work or school and thus their sizes and ratios are of interest to us.

For the purpose of clarity and further progress in the problem, let us give symbolic representation to the sides of the A4 paper. Note that A4 is chosen because it is the basis of all the other larger paper sizes in the A family, and the larger sizes are simply enlargements of the A4 paper size. This will make the labelling for larger paper sizes easier, as they can simply be expressed as multiples of the original A4 paper size.

Let the width of the A4 paper = w
Let the length of the A4 paper = l

Similar to Alison's diagram, I created the following diagram on my computer:

Image 1.0: *Diagrammatic Representation of the A family paper sizes*



(I included the A0 size also as it will be needed to solve the other parts of this problem).

As it can be seen from the above diagram, the lengths and widths of larger paper sizes can simply be expressed as multiples of the A4 paper size.

For example, the A1 paper size has twice the length of A4 ($2l$) and four times the width of A4 ($w + w + 2w = 4w$), so its dimensions are length $= 2l$ and width $= 4w$. This will ease the mathematical working as we will have to deal with less number of variables.

It can also be seen from the diagram that each successive paper size A(N-1) has twice the area than A(N). For example, the A3 paper has twice the area as A4. Geometrically, this holds true, but how does it fair algebraically? I was curious to find out:

The algebraic interpretation of the geometrical observation I have made is as follows:

$$Area\ of\ A3\ paper = 2 \times Area\ of\ A4\ paper$$

Let us first consider the left hand side (LHS) of the above equation:

Using the diagram 1.0, the length of an A3 paper is l and the width is $2w$. As it is a rectangle, the area is the product of the length and the width.

$$\therefore LHS = Area\ of\ A3\ paper = l \times 2w = 2lw$$

Let us now consider the right hand side (RHS):

Using the diagram 1.0 again, the length of an A4 paper is l and the width is w. As it is also a rectangle, the area is the product of the length and the width. Remember that we are looking for twice the area of an A4 paper on the RHS.

$$RHS = 2 \times Area\ of\ A4\ paper = 2 \times l \times w = 2lw = LHS$$

Thus, my observation that the area of an A3 paper is twice the area of an A4 paper holds true. What I have done is a mathematical verification of an observation I made. This will be helpful later in the problem as we progress.

Finding the ratio of the shorter to the longer side of a piece of A paper

This is the first sub-part of our problem at hand and is indeed very interesting due to its nature. We have to find the ratio of the shorter side to the longer side of an A paper. Continuing from where Alison left (in the hint section), we have to work out an expression for l in terms of w in our case.

We know that the rectangles are similar; i.e. they are simply enlargements (or shrinkages) of one another. Thus, the ratio between their sides is constant. Considering this rule for the A4 and A3 papers, this implies that:

$$l : w = 2w : l$$

The ratio of the length (l) of the A4 paper to the width (w) of the A4 is equal to the length ($2w$) of the A3 paper to the width (l) of the A3 paper. A constant ratio implies a constant proportion between the two sides in consideration. The above relationship can thus also be expressed as follows:

$$\frac{l}{w} = \frac{2w}{l}$$

Solving for the length l by cross multiplying, we have that:

$$l^2 = 2w^2$$

Now to work out the expression in terms of l, we can take the square root of both the sides (as it will cancel out the square terms). We have something as follows:

$$l = \sqrt{2}w$$

Now you might be wondering that the solution should be $l = \pm\sqrt{2}w$ and I have certainly made a mistake. The opposite is true, however. If this had been a traditional algebraic problem without real world significance, you might be right in presenting two solutions. In this case however, the negative solution is extraneous. That is, it is not relevant to our exploration. Having a negative ratio does not make sense here, as we do not have negative lengths or widths. If, however, this problem dealt with shapes on a coordinate plane, then we would present two solutions.

So we have completed our task of expressing l in terms of w and found out the ratio between the shorter side to the longer side. But what does it tell us? How is it helpful? What does it even mean? To answer these questions, the relationship we found simply tells us that any paper in the A family following the pattern as shown in Diagram 1.0 has this simple rule: the longer side is always the product of the square-root of two and the shorter side. Or in another words using decimals, the longer side is always approximately 1.41 times as big as the shorter side. Interesting no?

(Like we did a mathematical verification of our geometric observation, you can do a geometric verification of this mathematical derivation. Take a scale and measure the dimensions of an A family paper; does the above hold true? Or you can alternatively look up the sizes of different A family papers on the Internet and check the validity of our mathematical derivation)

Deducing the length and width of different A paper sizes

We are given that the area of an A0 paper sheet is 1 m^2 and we need to find out the lengths and widths of different A paper sizes. The problem might seem daunting at first – we are given *just the area* of 1 sheet of paper and we need to find the *dimensions* of *all* the other A paper sizes. But, the problem is rather simple as we work through it.

Coming back to our mathematical verification, we know that each successive paper size A(N-1) has twice the area than A(N). For example, the A3 paper has twice the area as A4. Using this fact, we can deduce the area of all the A paper sizes as follows:

$$\mathrm{A0} = 1\ m^2$$

Working backwards, as A0 has an area of one meter squared, A1 must have half the area as A0:

$$\mathrm{A1} = \frac{\mathrm{A0}}{2} = \frac{1}{2} m^2$$

In the same manner, we have that:

$$\mathrm{A2} = \frac{1}{4}\ m^2,\ \mathrm{A3} = \frac{1}{8}\ m^2, \mathrm{A4} = \frac{1}{16}\ m^2$$

Now, if we observe carefully, we can notice that the above terms follow a geometric sequence; a sequence where each successive term is a constant multiple of the previous term. In this case, the constant multiple or common ratio is $\frac{1}{2}$, as each successive term is half of the previous term. This can be defined using an infinite, converging recursive sequence, a sequence with infinitely many terms. This is analogous to *folding* the papers into 'half', where the total area of all the papers will eventually approach a finite value or converge to that value.

Now, we have the areas of different A paper sizes, but how do we work out the dimensions of these sides? Let us start by what we know and what we have, and let us take the example of the A4 paper:

What we know: The area of an A4 paper is $\frac{1}{16}\ m^2$ and is the product of its length and width. Thus, we have the following:

$$l \times w = \frac{1}{16}$$

Recall from the previous section that we have already established a relationship between l and w ($l = \sqrt{2}w$), and hence we now have a system of equations with two equations and two unknowns. Replacing the relationship from the previous part into the above equation, we have that:

$$\sqrt{2}w \times w = \frac{1}{16}$$

$$\therefore w^2 = \frac{\sqrt{2}}{32}$$

$\therefore w = 0.210\ m$ (3 significant digits)

Therefore, the width of an A4 paper is approximately 0.210 m. Let us now replace this back into the relationship between the length and the width to calculate the length.

$$l = \sqrt{2} \times w = \sqrt{2} \times 0.210$$

$$\therefore l = 0.297\ m$$

Therefore, the length of an A4 paper is approximately 0.297 m. Now, the lengths and widths of the remaining A paper sizes can easily be calculated as they are simply the multiples of the A4 size. For completion, they are listed here below:

A3

Length = 0.420 m, Width: 0.297 m

A2

Length = 0.594 m, Width = 0.420 m

A1

Length = 0.840 m, Width = 0.594 m

A0

Length = 1.189 m, Width = 0.840 m

On a photocopier, approximately what percentage would you need to scale by in order to photocopy an A3 poster onto A4 paper?

This is also an interesting sub-part linked to a real-life situation that may arise. If we want to print an A3 poster onto an A4 paper, by what factor do we scale? From earlier parts of this problem, we know that the A3 paper has twice the area as A4 or in other words, an A4 paper is half of an A3 paper (recall the observation we made using Diagram 1.0).

Therefore, the content on the A3 paper must be halved in order to fit it onto an A4 paper. This can be done by multiplying the A3 paper with 0.5 or scaling by 50%.

Finding a consistent way to define A(-1) and other negative paper sizes

Assuming that the negative paper sizes also follow the same geometric pattern as depicted in Diagram 1.0 and discussed in the earlier part of the problem, the negative paper sizes can also be described by a recursive sequence (recall the sequence with infinite terms).

Let the area of A family paper sizes = A_i, where i is the number in the family, eg. The area of an A4 paper will be A_4.

Using our previous deductions, we know that:

$$A_1 = \frac{1}{2}A_0 \iff A_0 = 2A_1$$

We know that each successive paper size A_{i-1} has twice the area than A_i. Using this logic and our assumption that negative paper sizes follow this pattern, we can state that A(-1) must have twice the area of A0. Therefore, we have that:

$$A_{-1} = 2A_0$$

Similarly, we have that:

$$A_{-2} = 2A_{-1}$$

$$A_{-3} = 2A_{-2}$$

$$A_{-4} = 2A_{-3}$$

and so on….

This is also a geometric sequence but with a common ratio of 2 (as each successive term is twice of the previous term). Interestingly, this is also an infinite sequence (because every large number is a multiple of 2 in this case) but is different from the infinite sequence which represents the area of the positive A paper sizes, as it will not converge to a finite sum. This is because we are doubling the area each time and hence increasing the total area of the papers each time we add another paper.

We thus have a consistent way of defining the negative paper sizes: the geometric sequence with first term 2 (area of A_{-1}) and common ratio 2.