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Exploring Cubic Functions

In the GeoGebra applet below, move the slider to change the function.
What stays the same? What changes?

Similarities between all the 11 cubic graphs as you move the slider to change 2 out of 3 of the roots of the cubic include the fact that they all pass through the origin (0,0) and therefore all have a y-intercept of 0, they are all positive cubics and the sum of their three roots is 0. In addition, where $1 \leq a \leq 10$ on the slider, the cubic graphs have turning points known as local maxima and local minima areas. Furthermore, all these cubic graphs are rotationally symmetric given that the maximum point is rotated to become a minimum and vice versa thus making the centre of rotation the midpoint of the line connecting the maximum and minimum points.
Differences between the cubic graphs include their different roots and the graph of $y = x^3$, occurring when $a = 0$ on the slider, has stationary point known as a point of inflection at $x = 0$ rather than local maxima and local minima areas which are present in the other ten cubic graphs.

Can you take a guess at what function is being graphed each time?

In order to determine or "guess" the function being graphed, you must determine the three roots of the cubic, which in this instance are always $x = 0$, $x = a$ and $x = -a$.Therefore, the function being graphed takes the form of $f(x) = x(x-a)(x+a)$.

Here is a second family of functions to explore. In this case, the three roots sum to zero.
What do you notice about the graphs and their functions?

The function for all these cubic graphs takes the form of $f(x) = (x-p)(x-q)(x-(p+q))$

What do they have in common?

The roots of all these cubic graphs, $x = p$, $x = q$ and $x = p+q$ all sum to zero and all these cubics are rotationally symmetric.

Can you explain any of the patterns that you notice?

A cubic function or a third-degree polynomial function is one that can be written in the form $f(x) = ax^3 + bx^2 + cx + d$ and all cubic graphs of this general cubic polynomial function have rotational symmetry about the origin given that $f(-x) = -f(x)$. This can be demonstrated through the use of calculus:
Given that $f(x) = ax^3 + bx^2 + cx + d$

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$f'(x) = 3ax^2 + 2bx + c$ (constant terms, in this case d, disappear on differentiation)

Stationary points occur when the first derivative , $f'(x) = 0$ which can also be written as $0 = 3ax^2 + 2bx + c$.

Using the quadratic formula, we can establish the two possible solutions for x:

$$x = \frac{-b+\sqrt{b^2-3ac}}{3a} \text{ or } x = \frac{-b-\sqrt{b^2-3ac}}{3a}$$

We know that the centre of rotation of the cubic polynomial is the midpoint of the line connecting the maximum and minimum points. Therefore, determination of the x and y coordinates of the midpoint gives:

$$x = \frac{(\frac{-b+\sqrt{b^2-3ac}}{3a}+\frac{-b-\sqrt{b^2-3ac}}{3a})}{2}$$

Therefore $x = \frac{-b}{3a}$

To determine the y coordinate of the midpoint, we must find half the sum of the values produced when the two solutions for x are plugged back into the original cubic function:]

$$y_1 = a(\frac{-b+\sqrt{b^2-3ac}}{3a})^3 + b(\frac{-b+\sqrt{b^2-3ac}}{3a})^2 + c(\frac{-b+\sqrt{b^2-3ac}}{3a}) + d$$

$$y_1 =- \frac{9abc-2b^3+2b^2\sqrt{-3ac+b^2}-6ac\sqrt{-3ac+b^2}}{27a^2} + d$$

$$y_2 = a(\frac{-b-\sqrt{b^2-3ac}}{3a})^3 + b(\frac{-b-\sqrt{b^2-3ac}}{3a})^2 + c(\frac{-b-\sqrt{b^2-3ac}}{3a}) + d$$

$$y_2 =- \frac{9abc-2b^3-2b^2\sqrt{-3ac+b^2}+6ac\sqrt{-3ac+b^2}}{27a^2} + d$$

The y coordinate of the midpoint of the lines is given by $\frac{y_1+y_2}{2}$:

$$\frac{y_1+y_2}{2} = \frac{\frac{-2(9abc-27a^2d-2b^3}{27a^2}}{2}$$

Thus the y coordinate of the midpoint is $\frac{-9abc+27a^2d+2b^3}{27a^2}$

We must translate the general cubic polynomial function such that the midpoint becomes the origin of the graph - we do this because as all cubic graphs of the general cubic polynomial function have rotational symmetry about the origin ,given that $f(-x) =- f(x)$, the midpoint of the line connecting the maximum and minimum points of the cubic represents the point of symmetry for the function presented.

The translation equations of the function for moving of the vertical and horizontal axes to render the midpoint as the origin of the graph are:

$X = x + \frac{b}{3a}$ and $Y = y - \frac{-9abc+27a^2d+2b^3}{27a^2}$ where X and Y represent the horizontal and vertical axes of the graph.

Substitution of $x = X - \frac{b}{3a}$ into the original function $f(x) = ax^3 + bx^2 + cx + d$ results in:

$$y = a(X - \frac{b}{3a})^3 + b(X - \frac{b}{3a})^2 + c(X - \frac{b}{3a}) + d$$

Therefore,

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$$g(X) = aX^3 + \left(\frac{3ac-b^2}{3a}\right)X$$

Thus,

$g(-X) = -g(X)$ and so all graphs of the general cubic function $f(x) = ax^3 + bx^2 + cx + d$ have rotational symmetry about the origin.

Here are six graphs and their functions. **Can you match the functions to their graphs?**

A is (3) or A is $y = x^3 - 7x^2 + 4x + 12$

B is (1) or B is $y = x^3 + 2x^2 - 5x + 12$

C is (4) or C is $y = x^3 - 3x^2 - x - 12$

D is (6) or D is $y = x^3 - x^2 - 8x + 12$

E is (2) or E is $y = x^3 + 3x^2 - 4x - 12$

F is (5) or F is $y = x^3 + x^2 - 8x - 12$

The illustration below shows the graphs of fourteen functions.
Two of them have equations $y = (x+6)^3 - 2$ and $y = -(x-9)^3 + 3$

Can you find the equations of the other twelve graphs in this pattern?

The equations of the other twelve graphs are:

$y = (x+9)^3 - 3$

$y = -(x+9)^3 - 3$

$y = -(x+6)^3 - 2$

$y = (x+3)^3 - 1$

$y = -(x+3)^3 - 1$

$y = x^3$

$y = -x^3$

$y = (x-3)^3 + 1$

$y = -(x-3)^3 + 1$

$y = (x-6)^3 + 2$

$y = -(x-6)^3 + 2$

$y = (x-9)^3 + 3$