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Summing Geometric Progressions

Series #1:

$$3 + 9 + 27 + 81 + 243 + \cdots + 15th\ term = \sum_{n=1}^{15} 3^n$$

We can easily see that the common ratio of this geometric series is $r = 3$ with 3 as its first term as well. Having said that we can rewrite the above series explicitly in powers of 3 and call it S:

$$S = 3 + 3^2 + 3^3 + \cdots + 3^{15}$$

Multiplying S by 3 gives:

$$3S = 3^2 + 3^3 + 3^4 \ldots + 3^{16}$$

Subtracting S from $3S$:

$$3S - S = 3^2 + 3^3 + 3^4 \ldots + 3^{16} - (3 + 3^2 + 3^3 + \cdots + 3^{15})$$

Dividing by 2:

$$2S = 3^{16} - 3$$

$$\boldsymbol{S = \frac{3^{16} - 3}{2}}$$

Series #2:

$$5 + 10 + 20 + 40 + 80 + \cdots + 12th\ term = \sum_{k=1}^{12} 5(2^{k-1})$$

This geometric series has first term 5 and a common ratio equal to 2. We can equivalently rewrite the above series as a product of 5 and a power of 2:

$$S = 5 + 5 \cdot 2^1 + 5 \cdot 2^2 + \cdots + 5 \cdot 2^{11}$$

Multiplying S by 2 gives:

$$2S = 5 \cdot 2^1 + 5 \cdot 2^2 + 5 \cdot 2^3 + \cdots + 5 \cdot 2^{12}$$

Subtracting S from $2S$:

$$2S - S = 5 \cdot 2^1 + 5 \cdot 2^2 + 5 \cdot 2^3 + \cdots + 5 \cdot 2^{12} - (5 + 5 \cdot 2^1 + 5 \cdot 2^2 + \cdots + 5 \cdot 2^{11})$$

So we get:

$$\boldsymbol{S = 5 \cdot 2^{12} - 5}$$

Series #3:

$$3+6+12+24+48+\cdots+20th\ term=\sum_{i=1}^{20}3\left(2^{i-1}\right)$$

This series has a first term equal to 3 and a common ratio equal to 2. Rewriting this series solely in terms of its first term and common ratio we obtain:

$$S=3+3\cdot 2^1+3\cdot 2^2+\cdots+3\cdot 2^{19}$$

Multiplying S by 2 gives:

$$2S=3\cdot 2^1+3\cdot 2^2+3\cdot 2^3+\cdots+3\cdot 2^{20}$$

Subtracting S from $2S$:

$$2S-S=3\cdot 2^1+3\cdot 2^2+3\cdot 2^3+\cdots+3\cdot 2^{20}-(3+3\cdot 2^1+3\cdot 2^2+\cdots+3\cdot 2^{19})$$

Simplifying the terms we get:

$$\boldsymbol{S=3\cdot 2^{20}-3}$$

Series #4:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots+10th\ term=\sum_{n=1}^{10}\frac{1}{2^n}$$

This geometric series has $\frac{1}{2}$ as its first term and its common ratio. This being said, we may rewrite the above series completely in terms of powers of $\frac{1}{2}$:

$$S=\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{10}$$

Multiplying S by $\frac{1}{2}$:

$$\frac{1}{2}S=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{11}$$

Subtracting S from $\frac{1}{2}S$:

$$\frac{1}{2}S-S=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{11}-\left(\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{10}\right)$$

$$-\frac{1}{2}S=\left(\frac{1}{2}\right)^{11}-\frac{1}{2}$$

Dividing by $-\frac{1}{2}$:

$$S=\frac{\left(\frac{1}{2}\right)^{11}-\frac{1}{2}}{-\frac{1}{2}}$$

Simplifying:

$$\boldsymbol{S = 1 - \left(\frac{1}{2}\right)^{10}}$$

General geometric series:

Consider the following general geometric series with first term a and common ratio r, where $a, r \in \mathbb{R}$, $r \neq 1$ and $n \in \mathbb{Z}^+$:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{j=0}^{n-1} ar^j$$

where S_n is the sum of the first n terms of the geometric series.

After determining the sums of the four previous series, we see that generic method is to multiply the series by its common ratio and then subtract the original from this new series. Applying this method, we multiply the above generic series by its common ratio:

$$S_n \cdot r = ar + ar^2 + \cdots + ar^n$$

Proceeding with the usual subtraction:

$$S_n \cdot r - S_n = ar + ar^2 + \cdots + ar^n - (a + ar + ar^2 + \cdots + ar^{n-1})$$

$$S_n \cdot r - S_n = ar^n - a$$

Factoring out S_n and a:

$$S_n(r - 1) = a(r^n - 1)$$

Dividing both sides by $r - 1$:

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$$

Extension to infinite geometric series:

We can define an infinite geometric series in a similar way:

$$S_\infty = a + ar + ar^2 + \cdots = \sum_{j=0}^{\infty} ar^j$$

where the first term is a and the common ratio is r, where $a, r \in \mathbb{R}$ and $r \neq 1$. An expression for the infinite sum can be found through two main ways: limits or Alison´s method.

Limit method:

We can use the general expression for the sum of the first n terms of a geometric series that we obtained previously and evaluate it as n approaches positive infinity. This makes sense because in an infinite geometric series, there are infinitely many terms so n is represented as being infinite.

$$S_\infty = \lim_{n\to\infty} \frac{a(r^n - 1)}{r - 1}$$

Depending on the value of r, the above limit will differ. Recalling from before, it was determined that $r \neq 1$. So we are left to analyze the case when $|r| < 1$ and when $|r| > 1$.

When $|r| < 1$:

$$S_\infty = \lim_{n\to\infty} \frac{a(r^n - 1)}{r - 1}$$

Factoring the constant part and manipulating the limit:

$$S_\infty = \frac{a}{r-1}\left[\lim_{n\to\infty}(r^n - 1)\right]$$

$$S_\infty = \frac{a}{r-1}\left[\lim_{n\to\infty} r^n - \lim_{n\to\infty} 1\right]$$

Since $\lim_{n\to\infty} r^n$=0 we get:

$$S_\infty = \frac{a}{r-1}[-1]$$

$$\therefore \boldsymbol{S_\infty = \frac{a}{1-r}}$$

When $|r| > 1$:

$$S_\infty = \frac{a}{r-1}\left[\lim_{n\to\infty} r^n - \lim_{n\to\infty} 1\right]$$

Since Since $\lim_{n\to\infty} r^n = \infty$ we get:

$$S_\infty = \frac{a}{r-1}[\infty - 1] = \infty$$

So the series is said to diverge in this case.

Therefore, we conclude that for an infinite geometric series to have a finite sum, the following condition must be satisfied: $|r| < 1$.

Alison´s method:

Starting with:

$$S_\infty = a + ar + ar^2 + \cdots$$

We then multiply S_∞ by r:

$$S_\infty \cdot r = ar + ar^2 + ar^3 \dots$$

After the usual subtraction:

$$S_\infty \cdot r - S_\infty = ar + ar^2 + ar^3 \dots - (a + ar + ar^2 + \cdots)$$

$$S_\infty \cdot r - S_\infty = -a$$

Isolating S_∞:

$$S_\infty (r - 1) = -a$$

$$S_\infty = \frac{-a}{r-1} = \frac{a}{1-r}$$

The condition for which this expression is finite was found previously by the limit method.