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### Hyperbolic Thinking

Steve believes that the functions shown below,  $A(x)$  and  $B(x)$ , are similar in some form to the sine and cosine trigonometric functions.

$$A(x) = \frac{1}{2}(10^x + 10^{-x}) \quad B(x) = \frac{1}{2}(10^x - 10^{-x})$$

To analyze this claim, let us begin by first recalling a few of the main properties regarding the sine and cosine functions and then proceeding by comparing their applicability to the functions  $A(x)$  and  $B(x)$ .

#### 1) $\sin^2 x + \cos^2 x = 1$

To apply this property to Steve's functions, it is useful to consider both  $A^2(x)$  and  $B^2(x)$  and then attempt to derive a relation.

$$A^2(x) = \frac{1}{4}(10^{2x} + 2 + 10^{-2x})$$

$$B^2(x) = \frac{1}{4}(10^{2x} - 2 + 10^{-2x})$$

From which we deduce the following:

$$\begin{aligned} A^2(x) - B^2(x) &= \frac{1}{4}(10^{2x} + 2 + 10^{-2x}) - \frac{1}{4}(10^{2x} - 2 + 10^{-2x}) \\ &= \frac{1}{4}(10^{2x} + 10^{-2x}) - \frac{1}{4}(10^{2x} + 10^{-2x}) + \frac{1}{4}(2) + \frac{1}{4}(2) = 1 \end{aligned}$$

We deduce the following property:

$$A^2(x) - B^2(x) = 1$$

From this analysis, we spot a fundamental difference between the way in which the cosine and sine functions are defined and the way in which functions A and B are defined. Mainly, if we let  $x = A(x)$  and  $y = B(x)$  we see that we are left with the equation of the unit hyperbola  $x^2 - y^2 = 1$ . This means that we may define a point on this unit hyperbola by the coordinates  $(A(x), B(x))$ , unlike the cosine and sine functions that are defined by the unit circle  $x^2 + y^2 = 1$ .

#### 2) Cosine is an even function and Sine is an odd function

Recalling that a function  $h(x)$  is even if it satisfies  $h(x) = h(-x)$  and odd if it satisfies  $-h(x) = h(-x)$ , let us apply these concepts to the functions A and B.

$$A(-x) = \frac{1}{2}(10^{-x} + 10^x) = A(x)$$

So we verify that function A is even.

$$B(-x) = \frac{1}{2}(10^{-x} - 10^x) = -B(x)$$

So function B is odd. Function A therefore presents the same type of symmetry that cosine presents and Function B has the same symmetry as the sine function.

#### 3) $\sin(2x) = 2 \sin x \cos x$

To compare this identity, let us first evaluate the product  $2A(x)B(x)$ :

$$2A(x)B(x) = 2 \left( \frac{1}{2}(10^x + 10^{-x}) \frac{1}{2}(10^x - 10^{-x}) \right) = \frac{1}{2}(10^{2x} - 10^{-2x})$$

which equals  $A(2x)$ . So we conclude that a similar property holds

$$A(2x) = 2A(x)B(x).$$

$$4) \cos(2x) = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x = \cos^2 x - \sin^2 x$$

Let us begin by obtaining expressions for  $2A^2(x)$  and  $2B^2(x)$ .

$$2A^2(x) = 2 \left( \frac{1}{4} (10^{2x} + 2 + 10^{-2x}) \right) = \frac{1}{2} (10^{2x} + 10^{-2x}) + 1$$

From which we can conclude that  $B(2x) = 2A^2(x) - 1$ .

$$2B^2(x) = 2 \left( \frac{1}{4} (10^{2x} - 2 + 10^{-2x}) \right) = \frac{1}{2} (10^{2x} + 10^{-2x}) - 1$$

From which we can conclude that  $B(2x) = 2B^2(x) + 1$ .

We can obtain yet another equation if we add both of these newly found equations:

$$2B(2x) = 2A^2(x) + 2B^2(x)$$

$$B(2x) = A^2(x) + B^2(x)$$

So there is a slight difference in signs.

$$5) \sin(x+y) = \sin x \cos y + \cos x \sin y, \cos(x+y) = \cos x \cos y - \sin x \sin y$$

Let us first obtain expressions for  $A(x)A(y) + B(x)B(y)$  and  $B(x)A(y) + A(x)B(y)$  and then see how these relate to  $A(x+y)$  and  $B(x+y)$ .

$$\begin{aligned} A(x)A(y) + B(x)B(y) &= \frac{1}{2} (10^x + 10^{-x}) \cdot \frac{1}{2} (10^y + 10^{-y}) + \frac{1}{2} (10^x - 10^{-x}) \cdot \frac{1}{2} (10^y - 10^{-y}) \\ &= \frac{1}{4} (10^{x+y} + 10^{x-y} + 10^{-x+y} + 10^{-x-y}) + \frac{1}{4} (10^{x+y} - 10^{x-y} - 10^{-x+y} + 10^{-x-y}) \\ &= \frac{1}{4} (10^{x+y} + 10^{-x-y}) + \frac{1}{4} (10^{x+y} + 10^{-x-y}) + \frac{1}{4} (10^{x-y} + 10^{-x+y}) - \frac{1}{4} (10^{x-y} + 10^{-x+y}) \\ &= \frac{1}{2} (10^{x+y} + 10^{-x-y}) \end{aligned}$$

Which happens to equal  $A(x+y)$ . So we have that  $A(x+y) = A(x)A(y) + B(x)B(y)$ .

$$\begin{aligned} B(x)A(y) + A(x)B(y) &= \frac{1}{2} (10^x - 10^{-x}) \cdot \frac{1}{2} (10^y + 10^{-y}) + \frac{1}{2} (10^x + 10^{-x}) \cdot \frac{1}{2} (10^y - 10^{-y}) \\ &= \frac{1}{4} (10^{x+y} + 10^{x-y} - 10^{-x+y} - 10^{-x-y}) + \frac{1}{4} (10^{x+y} - 10^{x-y} + 10^{-x+y} - 10^{-x-y}) \\ &= \frac{1}{4} (10^{x+y} - 10^{-x-y}) + \frac{1}{4} (10^{x+y} - 10^{-x-y}) + \frac{1}{4} (10^{x-y} - 10^{-x+y}) - \frac{1}{4} (10^{x-y} - 10^{-x+y}) \\ &= \frac{1}{2} (10^{x+y} - 10^{-x-y}) \end{aligned}$$

Which happens to equal  $B(x+y)$ . So we have that  $B(x+y) = B(x)A(y) + A(x)B(y)$ .

So both functions A and B more closely follow the sum formula of the sine function.

$$6) \frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(\cos x) = -\sin x$$

To calculate the derivatives for the functions A and B we recall that  $\frac{d}{dx}(a^x) = \ln a \cdot a^x$  where  $a \in \mathbb{R}^+$ . So with this in mind we first evaluate the derivative of the function A.

$$\begin{aligned} \frac{d}{dx} \left( \frac{1}{2} (10^x + 10^{-x}) \right) &= \frac{1}{2} \cdot \ln 10 \cdot 10^x + \frac{1}{2} \cdot (-1) \cdot \ln 10 \cdot 10^{-x} \\ &= \ln 10 \left( \frac{1}{2} (10^x - 10^{-x}) \right) \end{aligned}$$

So we find that  $\frac{d}{dx}(A(x)) = \ln 10 \cdot B(x)$ .

Now we calculate the derivative for the function B.

$$\begin{aligned} \frac{d}{dx} \left( \frac{1}{2} (10^x - 10^{-x}) \right) &= \frac{1}{2} \cdot \ln 10 \cdot 10^x - \frac{1}{2} \cdot (-1) \cdot \ln 10 \cdot 10^{-x} \\ &= \ln 10 \left( \frac{1}{2} (10^x + 10^{-x}) \right) \end{aligned}$$

So we find that  $\frac{d}{dx}(B(x)) = \ln 10 \cdot A(x)$ . So we find that both derivatives have a constant factor of  $\ln 10$ , rather than having a  $\pm 1$  for the case of the sine and cosine functions. Functions A and B are still similar to sine and cosine because if we differentiate A we get a relation with B and vice-versa.

**7)  $|\sin x| \leq 1$  and  $|\cos x| \leq 1$ .**

Property 7 is basically stating that the maximum value of the cosine and sine functions is 1. To analyze this property for the functions A and B, it is useful to calculate the stationary points and classify these either as maxima, minima or neither. To find these points, we set their derivatives equal to zero and solve. Beginning with the function A,

$$\begin{aligned} \ln 10 \cdot B(x) &= 0 \\ \ln 10 \left( \frac{1}{2} (10^x - 10^{-x}) \right) &= 0 \\ \frac{1}{2} (10^x - 10^{-x}) &= 0 \\ 10^x - 10^{-x} &= 0 \\ 10^x - \frac{1}{10^x} &= 0 \\ 10^{2x} - 1 &= 0 \\ (10^x - 1)(10^x + 1) &= 0 \end{aligned}$$

From which we can see that the only solution comes from the equation:

$$10^x - 1 = 0$$

So the only solution is  $x = 0$ . So the only stationary point has an x-coordinate of 0. To classify this point, we'll take a value less than 0 to check the monotonicity of A- whether it is increasing or decreasing. If x is -1 for instance, the derivative of A is approximately -11.4, so A is decreasing for values less than 0. For values greater than zero, say for instance x is 1, we have that the derivative is approximately 11.4 so the function A is increasing. We can therefore conclude that A has a minimum value at  $x = 0$ , which yields a value of 1. So we find that  $A(x) \geq 1$ . We will apply the same reasoning to the function B, beginning by setting its derivative equal to 0.

$$\begin{aligned} \ln 10 \cdot A(x) &= 0 \\ \ln 10 \left( \frac{1}{2} (10^x + 10^{-x}) \right) &= 0 \\ 10^x + 10^{-x} &= 0 \\ 10^{2x} + 1 &= 0 \end{aligned}$$

From which we conclude that there are no stationary points since the above equation has no real solutions. If we were to graph the function B on a graphic display calculator we would confirm this and see that  $B(x) \in ]-\infty, +\infty[$ . It is also important to note that function B is always increasing since  $\ln 10 \cdot A(x) > 0$ . So we see that property 7 does not apply at all to functions A and B.

Now that we have studied the main properties of sine and cosine and compared them to the functions A and B, we will study whether or not the relations derived above hold true for similar functions with any bases. So we will define the following two functions:

$f(x) = \frac{1}{2}(b^x + b^{-x})$  and  $g(x) = \frac{1}{2}(b^x - b^{-x})$  where  $b \in \mathbb{R}^+, b \neq 1$ . It is dispensable to go through all of the calculations again since properties analyzed

from numbers 1 to 5 are reliant upon algebraic rules such as  $b^n \cdot b^m = b^{n+m}$  and others that are not affected by a change of base. So they are base-independent. The same cannot be said about the last two properties. Let us calculate the derivatives of these newly defined functions.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}(b^x + b^{-x})\right) = \ln b \cdot g(x)$$

$$\frac{d}{dx}g(x) = \frac{d}{dx}\left(\frac{1}{2}(b^x - b^{-x})\right) = \ln b \cdot f(x)$$

We can also add both of these equations to obtain,

$$\frac{d}{dx}f(x) + \frac{d}{dx}g(x) = \ln b (f(x) + g(x))$$

so we see that property 6 is base-dependent. As such, this has some repercussions on property 7. The only stationary point still has an x-coordinate of 0 since it satisfies the equation,

$$b^x - 1 = 0$$

which would correspond to the only stationary point of the function  $f(x)$ , but since its monotonicity is still the same as that of the function A, it still has a minimum of 1 even though  $\ln b < 0$  for  $0 < b < 1$ . We still see that  $b^{2x} + 1 = 0$  has no solution so  $g(x)$ , much like function B, has no stationary points. However, it turns out that  $g(x)$  is always decreasing ( $\ln b \cdot f(x) < 0$ ) in the case that  $0 < b < 1$  since  $\ln b < 0$  but always increasing ( $\ln b \cdot f(x) > 0$ ) in the case that  $b > 1$ . So we see that the monotonicity of functions like  $g(x)$  is base-dependent as well.

After this analysis, a natural choice of base would be Euler's number,  $e$  which is 2.71828... This is due to the fact that it enable some of the base-dependent relations obtained to be simplified since  $\ln e = 1$ . If we use this base, we are then left with  $\frac{1}{2}(e^x + e^{-x})$  and  $\frac{1}{2}(e^x - e^{-x})$  which define the hyperbolic cosine and sine functions respectively.