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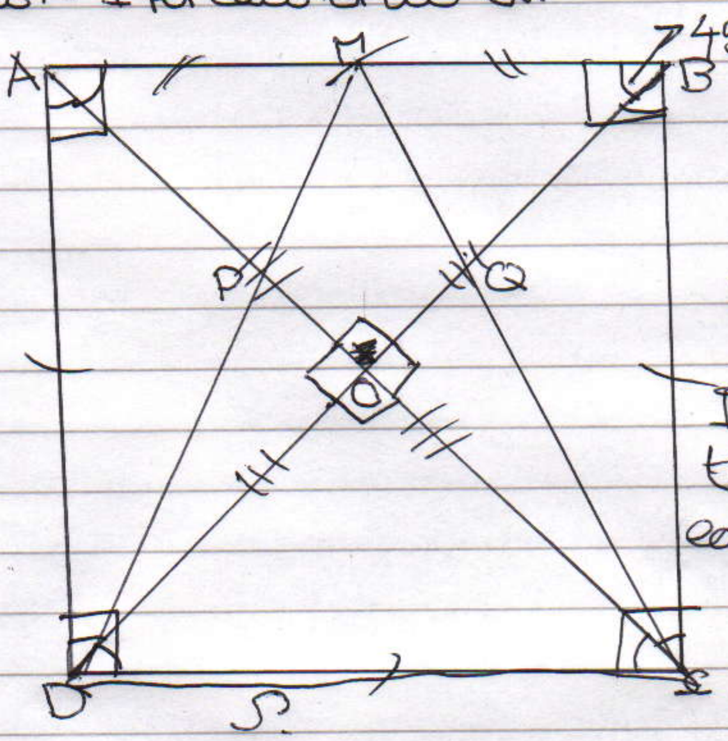
Kite Inside Square

When I first looked at this problem I decided to draw a diagram and mark all the pieces of information I know onto it.



Here, I construct points P, Q and O.

I also say that the length of each side is s

Now I am going to prove that $\triangle APM \cong \triangle BQM$.

No	Statement	Reason
i)	$\overline{AM} = \overline{AB}$	Midpoint of $\overline{AB}$ is M.
ii)	$\measuredangle PAM = \measuredangle QBM$	Both equal 45°.
iii)	$\measuredangle PMA = \measuredangle QMB$	$\measuredangle ADP = \measuredangle QCB$

$\therefore \triangle AMP \cong \triangle MQB$ QED

$\therefore MP = MQ$

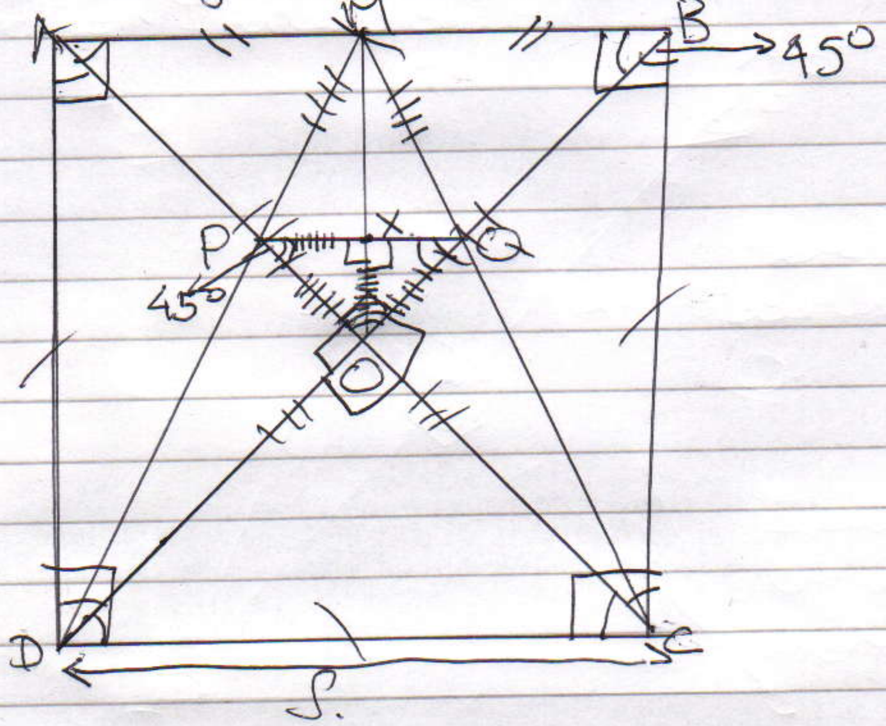
My next aim is to prove that $\triangle PDO = \triangle OQC$.

No	Statement	Reason
i)	$\measuredangle PQD = \measuredangle COQ$	Both are 90°

No	Statement	Reason
ii)	$\overline{DO} = \overline{CO}$	given from diagram.
iii)	$\overline{PD} = \overline{QC}$	$\overline{DP} = \overline{MD} - \overline{MP} = \overline{MC} - \overline{MQ} = \overline{QC}$

$\triangle PDO = \triangle CQC$ QED

$\therefore$ Our new diagram is:



I will prove that PX = OX.
I already know that $\measuredangle POQ$ is a right angle.
$\therefore \measuredangle OPQ = \measuredangle OQP = 45$.
As $\triangle POQ$ is an isosceles $\triangle$, O makes a right angle when it falls on X and bisects $\overline{PQ}$.
As $\measuredangle PXO = 90°$ and $\measuredangle POX = 45°$
$\therefore \measuredangle XPO = 45°$
$\Rightarrow \triangle PXO$ is isosceles.
$\Rightarrow \overline{PX} = \overline{XO}$.

The area of the kite =
$PX \times MX + PX \times OX$
$= PX(MX + OX)$
$= PX \times S/2$.

So in order to find the area, I need to find $\overline{PX}$, and in order to do that I need to find $\overline{PO}$

In order to find $\overline{PO}$, I do the following:
I will consider the triangle AMD. I will then find $\measuredangle ADM$.

I know that $\tan$ of $\measuredangle ADM = \dfrac{AM}{AD} = \dfrac{s/2}{s} = \dfrac{1}{2}$.

$\therefore \measuredangle ADM = \tan^{-1}\dfrac{1}{2}$.

$\Rightarrow \measuredangle PDO = 45 - \tan^{-1}\dfrac{1}{2}$.

$\therefore$ ~~tan(45-tan⁻¹~~

$\therefore \tan(45 - \tan^{-1}\tfrac{1}{2}) = \dfrac{PO}{DO}$

or $\tan(45 - \tan^{-1}\tfrac{1}{2}) = \dfrac{PO}{\sqrt{2}s} \times 2$.

or $\dfrac{\tan 45 - \tan(\tan^{-1}\frac{1}{2})}{1 + (\tan 45 \tan(\tan^{-1}\frac{1}{2})} = \dfrac{PO}{\sqrt{2}s} \times 2$.

or $\dfrac{1}{3} = \dfrac{PO}{\sqrt{2}s} \times 2$

or $\dfrac{\sqrt{2}s}{6} = PO$.

~~∴ PO=~~

And $PO^2 = 2PX^2$

or $\left(\dfrac{\sqrt{2}s}{6}\right)^2 = 2PX^2$

or $\dfrac{\sqrt{2}s}{6} = \sqrt{2}\,PX$

or $\dfrac{s}{6} = PX$.

$\therefore$ Area of $MPQO = \dfrac{s}{6} \times \dfrac{s}{2}$

or Area of $MPQO = \dfrac{s}{12}$

$\Rightarrow$ That the kite is $\dfrac{1}{12s}$ th of the square.

I am now going to order the statements for the solution that ~~was~~ uses coordinates:

1. G - Consider a unit square drawn on a coordinate grid.
2. D - The line joining (0,1) to (1,0) has the equation $y=1-x$.
3. B - The line joining (0,0) to $(\frac{1}{2},1)$ has the equation $y=2x$.
4. F - The point (a,b) is at the intersection of the lines $y=2x$ and $y=1-x$.
5. I - So $a=\frac{1}{3}$ and $b=\frac{2}{3}$.
6. A - the shaded area is made up of two congruent triangles, one of which has the vertices $(\frac{1}{3},\frac{2}{3})$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},1)$
7. H - The perpendicular of the triangle is $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.
8. C - Area of the triangle is $\frac{1}{2}(\frac{1}{2}\times\frac{1}{6})=\frac{1}{24}$
9. E - Therefore the shaded area is $2\times\frac{1}{24}=\frac{1}{12}$.

I will now order the the similar triangle statements.

1. F - Let ABCD be a unit square.
~~2. B - The line MF is half the length of AD.~~
2. A - As line AC intersects line MD at point E
3. C - Line AD is parallel to line MF, so ∡EDA and ∡EMF are equal and ∡EAD and ∡EFM are equal
4. D - Therefore, $\triangle AED$ and $\triangle FEM$ are similar
5. B - The line MF is half the length of AD.
6. E - Therefore, the line EH is half the length of PE
7. H - PH has length $\frac{1}{2}$ units so PE has length $\frac{1}{3}$ units and EH has length $\frac{1}{6}$ units.
8. I - MEF has area $\frac{1}{2}(\frac{1}{2}\times\frac{1}{6})=\frac{1}{24}$ sq units
9. G - Therefore the shaded region MEFG $=\frac{1}{24}\times 2=\frac{1}{12}$

I will now order the Pythagoras statements.

1. I - Assume that the sides of the square are each 2 units long. Thus DJ and FJ are each 1 unit long.

2. E - By Pythagoras, DF has length $\sqrt{2}$.

3. G - Area of DFE $= \frac{DF \times EF}{2} = \frac{\sqrt{2} \times EF}{2} = \frac{EF}{\sqrt{2}}$.

4. B - $(EH)^2 + (HF)^2 = (EF)^2$

$EH = HF$

$(EH)^2 = \frac{1}{2}(EF)^2$

$EH = \frac{1}{\sqrt{2}} EF$

5. D - Area of $\triangle$ MEF $= \frac{1}{2}(1 \times EH) = \frac{1}{2} \frac{EF}{\sqrt{2}}$

6. H - So the shaded area MEFG is equal to the area of $\triangle$DFE

7. A - The area of $\triangle$DMC = 2 squnits.
The area of $\triangle$DFC = 1 squnit
Thus the combined area of $\triangle$DFE, $\triangle$CFG and MEFG = 1 squnit.

8. C - Areas of $\triangle$DFE, $\triangle$CFG and MEFG are equal, so each has an area of $\frac{1}{3}$ squnits.

9. F - The total area of the square is 4 squnits, so the shaded area is $\frac{1}{12}$th the area of the whole square.